WIJSMAN $\mathcal{I}$-INVARIANT CONVERGENCE OF SEQUENCES OF SETS

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Abstract

In this paper, we study the concepts of Wijsman $\mathcal{I}$-invariant convergence ($\mathcal{I}_\sigma^W$), Wijsman $\mathcal{I}^*$-invariant convergence ($\mathcal{I}_\sigma^{*W}$), Wijsman p-strongly invariant convergence ($[WV_\sigma]_p$) of sequences of sets and investigate the relationships among Wijsman invariant convergence, $[WV_\sigma]_p$, $\mathcal{I}_\sigma^W$ and $\mathcal{I}_\sigma^{*W}$. Also, we introduce the concepts of $\mathcal{I}_\sigma^W$-Cauchy sequence and $\mathcal{I}_\sigma^{*W}$-Cauchy sequence of sets.

Introduction

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [5], Schoenberg [22] and studied by many authors. Nuray and Ruckle [14] indepedently introduced the same with another name generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [7] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$.

Introduction

Nuray and Rhoades [13] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [29] defined the Wijsman lacunary statistical convergence of set sequences and considered its relation with Wijsman statistical convergence defined by Nuray and Rhoades. Kişi and Nuray [6] introduced a new convergence notion, for sequence of sets called Wijsman $\mathcal{I}$-convergence. The concept of convergence of sequence of numbers has been extended by several authors to convergence of set sequences (see, [1, 2, 3, 23, 27, 28, 30, 32, 33]).

Introduction

Several authors including Raimi [20], Schaefer [21], Mursaleen [11], Savaș [24], Pancaroğlu and Nuray [18] and some authors have studied invariant convergent sequences. Nuray et al. [16] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. The concept of strongly σ-convergence was defined by Mursaleen [10]. Savaș and Nuray [26] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Recently, the concept of strong σ-convergence was generalized by Savaș [24]. Nuray and Ulusu [17] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers.

Introduction

In this paper, we study the concepts of Wijsman $\mathcal{I}$-invariant convergence ($\mathcal{I}_\sigma^W$), Wijsman $\mathcal{I}^*$-invariant convergence (${\mathcal{I}^*}_\sigma^W$), Wijsman p-strongly invariant convergence ($[WV_\sigma]_p$) and investigate the relationships among Wijsman invariant convergence, $[WV_\sigma]_p$, $\mathcal{I}_\sigma^W$ and ${\mathcal{I}^*}_\sigma^W$. Also, we introduce the concepts of $\mathcal{I}_\sigma$-Cauchy sequence and $\mathcal{I}_\sigma^*$-Cauchy sequence of sets.

Now, we recall the ideal convergence, invariant convergence, sequence of sets and basic definitions and concepts (See [7, 9, 13, 15, 16, 17, 18, 19, 20, 21, 32, 33]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definitions and Notations

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

Lemma 1 ([7])

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Definitions and Notations

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$. If $x = (x_k)$ is $\mathcal{I}$-convergent to L, then we write $\mathcal{I} - \lim x = L$.
Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1, 1, 1, ...)$, and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

Definitions and Notations

The mappings σ are one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [8].
It can be shown [25] that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Definitions and Notations

A bounded sequence $(x = x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m$$

and in this case, we write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.
In the case, $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

Definitions and Notations

The concept of strong σ-convergence was generalized by Savaș [24] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0 \text{ uniformly in n} \right\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.
A sequence $x = (x_k)$ is σ-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \left| \left\{ k \le m : |x_{\sigma^k(n)} - L| \ge \varepsilon \right\} \right| = 0, \text{ uniformly in n.}$$

In this case, we write $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$.

Definitions and Notations

Nuray et al. [16] introduced the concepts of σ-uniform density and $\mathcal{I}_\sigma$-convergence.
Let $A \subseteq \mathbb{N}$ and

$$s_n = \min_m \left| A \cap \left\{ \sigma(m), \sigma^2(m), ..., \sigma^n(m) \right\} \right|$$

and

$$S_n = \max_m \left| A \cap \left\{ \sigma(m), \sigma^2(m), ..., \sigma^n(m) \right\} \right|.$$

If the following limits exists

$$\underline{V}(A) = \lim_{n\to\infty} \frac{s_n}{n}, \qquad \overline{V}(A) = \lim_{n\to\infty} \frac{S_n}{n}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.
Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Definitions and Notations

A sequence (x_k) is said to be $\mathcal{I}_\sigma$-convergent to the number L if for every $\varepsilon > 0$,

$$A_\varepsilon = \left\{k : |x_k - L| \geq \varepsilon\right\} \in \mathcal{I}_\sigma,$$

that is, $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim x_k = L$.
For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definitions and Notations

Throughout the paper, we let (X, ρ) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be any non-empty closed subsets of X.

A sequence $\{A_k\}$ is Wijsman convergent to A if $\lim\limits_{k\to\infty} d(x, A_k) = d(x, A)$, for each $x \in X$. In this case, we write $W - \lim A_k = A$.

A sequence $\{A_k\}$ is bounded if $\sup\limits_{k} d(x, A_k) < \infty$, for each $x \in X$.

L_∞ denotes the set of bounded sequences of sets.

A sequence $\{A_k\}$ is said to be Wijsman invariant convergent to A if for each $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_{\sigma^k(m)}) = d(x, A), \text{ uniformly in } m.$$

Definitions and Notations

A sequence $\{A_k\}$ is said to be Wijsman strongly invariant convergent to A, if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)| = 0, \text{ uniformly in } m.$$

Definitions and Notations

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-convergent to A if for every $\varepsilon > 0$
$A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}$.
Let (X, ρ) be a separable metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. A sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that for each $x \in X$, $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$.

Definitions and Notations

A sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that $\{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \in \mathcal{I}$.
A sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that the subsequence $A_M = \{A_{m_k}\}$ is Wijsman Cauchy in X that is, $\lim_{k,p\to\infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0$.
An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{E_1, E_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{F_1, F_2, \cdots\}$ such that $E_j \Delta F_j$ is a finite set for $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}$.

Main Results

Definition 2

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant convergent or $\mathcal{I}_\sigma^W$-convergent to A if for every $\varepsilon > 0$, the set

$$A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\}$$

belongs to $\mathcal{I}_\sigma$, that is, $V(A(\varepsilon, x)) = 0$. In this case, we write $A_k \to A(\mathcal{I}_\sigma^W)$ and the set of all Wijsman $\mathcal{I}$-invariant convergent sequences of sets will be denoted $\mathcal{I}_\sigma^W$.

Main Results

Theorem 3

Let $\{A_k\}$ is bounded sequence. If $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A, then $\{A_k\}$ is Wijsman invariant convergent to A.

Main Results

Proof: Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. For each $x \in X$, we estimate

$$t(m,n,x) := \left| \frac{d(x, A_{\sigma(m)}) + d(x, A_{\sigma^2(m)}) + \cdots + d(x, A_{\sigma^n(m)})}{n} - d(x, A) \right|.$$

Then, for each $x \in X$ we have

$$t(m,n,x) \leq t^1(m,n,x) + t^2(m,n,x),$$

where

$$t^1(m,n,x) := \frac{1}{n} \sum_{\substack{j=1 \\ |d(x, A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|$$

and

$$t^2(m,n,x) := \frac{1}{} \sum^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|.$$

Main Results

Proof:

$$
\begin{aligned}
t^1(m,n,x) &\leq \frac{L}{n}|\{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\}| \\
&\leq L.\frac{\max_m |\{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\}|}{n} \\
&= L.\frac{S_n}{n}.
\end{aligned}
$$

Hence, $\{A_k\}$ is Wijsman invariant convergent to A.

Main Results

Definition 4

Let (X, ρ) be a separable metric space. The sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-invariant convergent or ${\mathcal{I}_\sigma^*}^W$-convergent to A if there exists a set $M = \{m_1 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that for each $x \in X$,

$$\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A).$$

Main Results

Theorem 5

If a sequence $\{A_k\}$ is $\mathcal{I}_\sigma^{*W}$-convergent to A, then this sequence is $\mathcal{I}_\sigma^W$-convergent to A.

Main Results

Proof: By assumption, there exists a set $H \in \mathcal{I}_\sigma$ such that for $M = \mathbb{N} \setminus H = \{m_1 < \cdots < m_k < \cdots\}$ we have

$$\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A), \tag{4.1}$$

for each $x \in X$. Let $\varepsilon > 0$ by (4.1), there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$,

$$|d(x, A_{m_k}) - d(x, A)| < \varepsilon,$$

for each $k > k_0$. Then, obviously

$$\{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}. \tag{4.2}$$

Since $\mathcal{I}_\sigma$ is admissible, the set on the right-hand side of (4.2) belongs to $\mathcal{I}_\sigma$. So $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A.

Main Results

Theorem 6

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP). If $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A, then $\{A_k\}$ is $\mathcal{I}_\sigma^{*W}$-convergent to A.

Main Results

Proof: Suppose that $\mathcal{I}_\sigma$ satisfies condition (AP). Let $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A. Then, for $\varepsilon > 0$ and for each $x \in X$

$$\{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_\sigma.$$

Put

$$E_1 = \{k : |d(x, A_k) - d(x, A)| \geq 1\} \;\; and \;\; E_n = \left\{k : \frac{1}{n} \leq |d(x, A_k) - d(x, A$$

for $n \geq 2$ and for each $x \in X$. Obviously $E_i \cap E_j = \emptyset$, for $i \neq j$. By condition (AP) there exists a sequence of $\{F_n\}_{n\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = (\bigcup_{j=1}^{\infty} F_j) \in \mathcal{I}_\sigma$. It is sufficient to prove that for $M = \mathbb{N} \setminus F$ and for each $x \in X$, we have

$$\lim_{k\to\infty} d(x, A_k) = d(x, A), \quad k \in M. \tag{4.3}$$

Main Results

Proof: Let $\lambda > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \lambda$. Then, for each $x \in X$,

$$\{k : |d(x, A_k) - d(x, A)| \geq \lambda\} \subset \bigcup_{j=1}^{n+1} E_j.$$

Since $E_j \Delta F_j$, $j = 1, 2, \ldots, n+1$ are finite sets, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{n+1} F_j\right) \cap \{k : k > k_0\} = \left(\bigcup_{j=1}^{n+1} E_j\right) \cap \{k : k > k_0\}. \tag{4.4}$$

If $k > k_0$ and $k \notin F$, then $k \notin \bigcup_{j=1}^{n+1} F_j$ and by (4.4) $k \notin \bigcup_{j=1}^{n+1} E_j$. But then

$$|d(x, A_k) - d(x, A)| < \frac{1}{n+1} < \lambda$$

Main Results

Definition 7

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant Cauchy sequence or $\mathcal{I}_\sigma^W$-Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon, x) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \Big\{k : |d(x, A_k) - d(x, A_N)| \geq \varepsilon\Big\} \in \mathcal{I}_\sigma,$$

that is, $V\big(A(\varepsilon, x)\big) = 0$.

Main Results

Definition 8

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}^*$-invariant Cauchy sequence or $\mathcal{I}^{*W}_{\sigma}$-Cauchy sequence if there exists a set $M = \{m_1 < \cdots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that

$$\lim_{k,p\to\infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0,$$

for each $x \in X$.

Main Results

We give following theorems which show relationships between $\mathcal{I}_\sigma^W$-convergence, $\mathcal{I}_\sigma^W$-Cauchy sequence and $\mathcal{I}^*{}_\sigma^W$-Cauchy sequence. The proof of them are similar to the proof of Theorems in [4, 12], so we omit them.

Theorem 9

If a sequence $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent, then $\{A_k\}$ is an $\mathcal{I}_\sigma^W$-Cauchy sequence.

Theorem 10

If a sequence $\{A_k\}$ is $\mathcal{I}^*{}_\sigma^W$-Cauchy sequence, then $\{A_k\}$ is $\mathcal{I}_\sigma^W$-Cauchy sequence.

Main Results

Theorem 11

Let $\mathcal{I}_\sigma$ has property (AP). Then the concepts $\mathcal{I}_\sigma^W$-Cauchy sequence and $\mathcal{I}^{*W}_\sigma$-Cauchy sequence coincides.

Definition 12

The sequence $\{A_k\}$ is said to be Wijsman p-strongly invariant convergent to A, if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)|^p = 0, \text{ uniformly in m},$$

where $0 < p < \infty$. In this case, we write $A_k \to A[WV_\sigma]_p$ and the set of all Wijsman p-strongly invariant convergent sequences of sets will be denoted $[WV_\sigma]_p$.

Main Results

Theorem 13

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ *be an admissible ideal and* $0 < p < \infty$.

(i) *If* $A_k \to A([WV_\sigma]_p)$, *then* $A_k \to A(\mathcal{I}_\sigma^W)$.

(ii) *If* $\{A_k\} \in L_\infty$ *and* $A_k \to A(\mathcal{I}_\sigma^W)$, *then* $A_k \to A([WV_\sigma]_p)$.

(iii) *If* $\{A_k\} \in L_\infty$, *then* $\{A_k\}$ *is* $\mathcal{I}_\sigma^W$*-convergent to* A *if and only if* $A_k \to A([WV_\sigma]_p)$.

Main Results

Proof: (i): If $A_k \to A([WV_\sigma]_p)$, then for $\varepsilon > 0$ and for each $x \in X$ we can write

$$\sum_{j=1}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|^p \geq \sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)})-d(x,A)|\geq\varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) - d(x$$

$$\geq \varepsilon^p |\{j \leq n : |d(x, A_{\sigma^j(m)}) - d(x, A)| \geq \varepsilon$$

$$\geq \varepsilon^p \max_m |\{j \leq n : |d(x, A_{\sigma^j(m)}) - d(x, A$$

and

$$\frac{1}{n} \sum_{j=1}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|^p \geq \varepsilon^p . \frac{\max_m |\{1 \leq j \leq n : |d(x, A_{\sigma^j(m)})}{n}$$

$$= \varepsilon^p \frac{S_n}{n}$$

Main Results

Proof: (ii): Suppose that $\{A_k\} \in L_\infty$ and $A_k \to A(\mathcal{I}_\sigma^W)$. Let $\varepsilon > 0$. By assumption we have $V(A_\varepsilon) = 0$. Since $\{A_k\}$ is bounded, $\{A_k\}$ implies that there exist $L > 0$ such that for each $x \in X$,

$$|d(x, A_{\sigma^j(m)}) - d(x, A)| \leq L,$$

for all j and m. Then, we have

$$\frac{1}{n}\sum_{j=1}^{n}|d(x, A_{\sigma^j(m)}) - d(x, A)|^p = \frac{1}{n}\sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)})-d(x,A)|\geq\varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) -$$

$$+ \frac{1}{n}\sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)})-d(x,A)|<\varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) -$$

$$\leq L \frac{\max_m |\{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) -$$

Main Results

Now, we shall state a theorem that gives a relationships betweeen WS_σ and $\mathcal{I}_\sigma^W$.

Theorem 14

A sequence $\{A_k\}$ is WS_σ-convergent to A if and only if it is $\mathcal{I}_\sigma^W$-convergent to A.

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