STRONGLY $\mathcal{I}_2$-LACUNARY CONVERGENCE AND $\mathcal{I}_2$-LACUNARY CAUCHY DOUBLE SEQUENCES OF SETS

ERDİNÇ DÜNDAR UĞUR ULUSU AND NİMET PANCAROĞLU

Abstract. In this paper, we study the concepts of Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationship between them.

1. INTRODUCTION

The concept of $\mathcal{I}$-convergence in a metric space, which is a generalized from of statistical convergence, was introduced by Kostyrko et al. [10]. Later it was further studied many others. Recently, Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Also, Yamancı and Gürdal [18] introduced the notion lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy in the topology induced by random n-normed spaces and prove some important results. Debnath [6] studied the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of ideal convergence. Tripathy et al. [13] introduced the concepts of $\mathcal{I}$-lacunary convergent sequences.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 4, 8, 11, 14, 16]). Nuray and Rhoades [11] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray [14] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Also, Ulusu and Nuray [15] introduced the concept of Wijsman strongly lacunary summability of sequences of sets. Recently, Kişi and Nuray [8] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Kişi et al. [9] defined Wijsman $\mathcal{I}$-asymptotically lacunary statistical equivalence of sequences of sets.

In this paper, we study the concepts of Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationship between them.

2. DEFINITIONS AND NOTATIONS

Let (X,ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x,A) = \inf_{a\in A} \rho(x,a).$$

2010 *Mathematics Subject Classification.* 40A05, 40A35.

Key words and phrases. Lacunary sequence, $\mathcal{I}_2$-convergence, Double sequence of sets, Wijsman convergence.

Definition 2.1. *([2]) Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k\to\infty} d(x,A_k) = d(x,A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 2.2. *([10]) A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if*

(i) $\emptyset \in \mathcal{I}$,

(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,

(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.3. *([10]) A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter if and only if*

(i) $\emptyset \notin F$,

(ii) For each $A, B \in F$ we have $A \cap B \in F$,

(iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

Proposition 2.4. *([10]) $\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

Definition 2.5. *([10]) An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.*

Definition 2.6. *([10]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. A sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set*

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$.

Definition 2.7. *([8]) Let (X,ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$ the set*

$$A(x,\varepsilon) = \{k \in \mathbb{N} : |d(x,A_k) - d(x,A)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. In this case we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$.

Definition 2.8. *([7]) For any lacunary sequence $\theta = \{k_r\}$, a sequence (x_k) of elements of $\mathbb{R}$ is said to be strongly lacunary convergent to $L \in \mathbb{R}$ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0.$$

Definition 2.9. *([14]) Let (X,ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x,A_k) - d(x,A)| = 0.$$

In this case we write $A_k \to A([WN]_\theta)$ or $A_k \overset{[WN]_\theta}{\to} A$.

Definition 2.10. *([9]) Let (X,ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to A or $N_\theta\left(\mathcal{I}_W\right)$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$, the set*

$$A(\varepsilon,x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |d(x,A_k) - d(x,A)| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(N_\theta\left[\mathcal{I}_W\right]\right)$.

Definition 2.11. *([8]) Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and (X,d) be a separable metric space. For any non-empty closed subsets A_n in X, we say that the sequence $\{A_n\}$ is Wijsman $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that the set*

$$A(x,\varepsilon) = \{n \in \mathbb{N} : |d(x,A_n) - d(x,A_N)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$.

3. MAIN RESULTS

Throughout the paper we take (X,ρ) be a separable metric space, $\theta = \{k_{rj}\}$ be a double lacunary sequence, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be non-empty closed subsets of X.

Definition 3.1. *The sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta\left(\mathcal{I}_{W_2}\right)$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$, the set*

$$A(\varepsilon,x) = \left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\left(N_\theta\left[\mathcal{I}_{W_2}\right]\right)$.

Theorem 3.2. *If $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A, then it is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A. For each $\varepsilon > 0$ and $x \in X$ there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon,$$

for all $k, j \geq k_0$. Then

$$T(x, \varepsilon) = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \subset \{1, 2, \cdots, k_0 - 1\}.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal we have $\{1, 2, \cdots, k_0 - 1\} \in \mathcal{I}_2$ and so $T(x, \varepsilon) \in \mathcal{I}_2$. This completes the proof. □

Definition 3.3. *The sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} d(x, A_{kj}) = d(x, A).$$

In this case, we write $A_{kj} \to A \left(N_\theta \left(\mathcal{I}_{W_2}^* \right) \right)$.

Definition 3.4. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case, we write $A_{kj} \to A \left(N_\theta \left[\mathcal{I}_{W_2}^* \right] \right)$.

Theorem 3.5. *If the sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A. Then, there exists a set $M = \{(k,j) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon,$$

for each $\varepsilon > 0$ and for all $k, j \geq k_0 = k_0(\varepsilon, x) \in \mathbb{N}$. Hence, for each $\varepsilon > 0$ and $x \in X$ we have

$$\begin{aligned} T(\varepsilon, x) &= \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \\ &\subset H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big), \end{aligned}$$

for $\mathbb{N} \times \mathbb{N} \backslash M' = H \in \mathcal{I}_2$. Since $\mathcal{I}_2$ is an admissible ideal we have

$$H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big) \in \mathcal{I}_2$$

and so $T(\varepsilon, x) \in \mathcal{I}_2$. Hence, this completes the proof. □

Theorem 3.6. *Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal with property $(AP2)$. If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then for each $\varepsilon > 0$ and $x \in X$

$$T(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Put

$$T_1 = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq 1 \right\},$$

$$T_p = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{V_p\}_{p \in \mathbb{N}}$ such that $T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each j and $V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}_2$. We prove that for each $x \in X$

$$\lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0,$$

for $M = \mathbb{N} \times \mathbb{N} \backslash V \in F(\mathcal{I}_2)$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then for each $x \in X$.

$$\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} T_j$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1, 2, \cdots, q-1\}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} T_j \right) \cap \{(k,j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\} = \left(\bigcup_{j=1}^{q-1} V_j \right) \cap \{(k,j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\}.$$

If $k, j \geq n_0$ and $(k,j) \notin V$, then

$$(k,j) \notin \bigcup_{j=1}^{q-1} V_j \ \text{ and so } \ (k,j) \notin \bigcup_{j=1}^{q-1} T_j.$$

Thus, for each $x \in X$ we have

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0.$$

Hence, for each $x \in X$ we have $A_{kj} \to A \left(N_\theta \left[\mathcal{I}^*_{W_2} \right] \right)$. This completes the proof. □

Definition 3.7. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists a number $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that*

$$A(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Theorem 3.8. *If $\{A_{kj}\}$ is Wijsman strongly lacunary Cauchy sequence, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. The proof is routine verification so we omit it. □

Theorem 3.9. *If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then for each $\varepsilon > 0$ and $x \in X$, we have

$$T\left(\frac{\varepsilon}{2}, x\right) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, the set

$$T^c\left(\frac{\varepsilon}{2}, x\right) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| < \frac{\varepsilon}{2}\right\}$$

is non-empty and belongs to $F(\mathcal{I}_2)$. So, we can choose positive integers r,u such that $(r,u) \notin T(\frac{\varepsilon}{2}, x)$, we have

$$\frac{1}{h_r\bar{h}_u}\sum_{(k_0,j_0)\in I_{ru}} |d(x,A_{k_0j_0}) - d(x,A)| < \frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon, x) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{k_0j_0})| \geq \varepsilon\right\}.$$

We show that $B(\varepsilon,x) \subset T(\frac{\varepsilon}{2}, x)$. Let $(r,u) \in B(\varepsilon,x)$ then we have

$$\begin{aligned}
\varepsilon &\leq \frac{1}{h_r\bar{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{k_0j_0})| \\
&\leq \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| + \frac{1}{h_r\bar{h}_u}\sum_{(k_0,j_0)\in I_{ru}} |d(x,A_{k_0j_0}) - d(x,A)| \\
&< \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| + \frac{\varepsilon}{2}.
\end{aligned}$$

This implies that

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| > \frac{\varepsilon}{2}$$

and therefore $(r,u) \in T(\frac{\varepsilon}{2}, x)$. Hence, we have $B(\varepsilon,x) \subset T(\frac{\varepsilon}{2}, x)$. This shows that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. □

Definition 3.10. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ and a number $N = N(\varepsilon,x) \in \mathbb{N}$ such that*

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{st})| < \varepsilon$$

for every $k,j,s,t \geq N$.

Theorem 3.11. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. Suppose that $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence. Then, for each $\varepsilon > 0$ and $x \in X$, there exists a set $M = \{(k,j) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ and a number $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j),(s,t) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| < \varepsilon$$

for every $k, j, s, t \geq k_0$.

Let $H = \mathbb{N} \times \mathbb{N} \backslash M'$. It is obvious that $H \in \mathcal{I}_2$ and

$$\begin{aligned} T(\varepsilon, x) &= \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j),(s,t) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| \geq \varepsilon \right\} \\ &\subset H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big). \end{aligned}$$

As $\mathcal{I}_2$ be a strongly admissible ideal then,

$$H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big) \in \mathcal{I}_2.$$

Therefore, we have $T(\varepsilon, x) \in \mathcal{I}_2$, that is, $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets. ☐

Combining Theorem 3.5 and Theorem 3.9, we have following Theorem:

Theorem 3.12. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Theorem 3.13. *If $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ is an admissible ideal with the property $(AP2)$ then the concepts Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy double sequence and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequence of sets coincide in X.*

Proof. If a sequence is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence, then it is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets by Theorem 3.6, where $\mathcal{I}$ need not have the property (AP).

Now, it is sufficient to prove that a sequence $\{A_k\}$ in X is a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence under assumption that it is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence. Let $\{A_k\}$ in X be a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence. Then, for each $\varepsilon > 0$ and $x \in X$, there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A_N)| \geq \varepsilon\} \in \mathcal{I}.$$

Let

$$P_i = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A_{n_i})| < \frac{1}{i} \right\}; \ (i = 1, 2, \ldots),$$

where $n_i = N(1 \backslash i)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$, $(i = 1, 2, \ldots)$. Since $\mathcal{I}$ has the property (AP), then by Lemma **??** there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$, and $P \backslash P_i$ is finite for all i.

Now, we show that

$$\lim_{\substack{k,n\to\infty \\ k,n\in P}} \frac{1}{h_r}\sum_{k,n\in I_r}|d(x,A_k)-d(x,A_n)|=0, \text{ for each } x \text{ in } X.$$

To prove this, let $\varepsilon>0$ and $j\in\mathbb{N}$ such that $j>2/\varepsilon$. If $k,n\in P$ then $P\backslash P_j$ is a finite set, so there exists $m=m(j)$ such that $k,n\in P_j$ for all $k,n>m(j)$. Therefore, for each x in X,

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,A_{n_j})|<\frac{1}{j} \;\; and \;\; \frac{1}{h_r}\sum_{n\in I_r}|d(x,A_n)-d(x,A_{n_j})|<\frac{1}{j},$$

for all $k,n>m(j)$. Hence, it follows that

$$\begin{aligned}\frac{1}{h_r}\sum_{k,n\in I_r}|d(x,A_k)-d(x,A_n)| &\leq \frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,A_{n_j})| \\ &+ \frac{1}{h_r}\sum_{n\in I_r}|d(x,A_n)-d(x,A_{n_j})| \\ &< \frac{1}{j}+\frac{1}{j}=\frac{2}{j} \\ &< \varepsilon,\end{aligned}$$

for all $k,n>m(j)$ and each x in X. Thus, for any $\varepsilon>0$ there exists $m=m(\varepsilon)$ such that for $k,n>m(\varepsilon)$ and $k,n\in P\in\mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r}\sum_{k,n\in I_r}|d(x,A_k)-d(x,A_n)|<\varepsilon, \text{ for each } x \text{ in } X.$$

This shows that the sequence $\{A_k\}$ in X is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence of sets.

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Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey

E-mail address: erdincdundar79@gmail.com, edundar@aku.edu.tr

E-mail address: ulusu@aku.edu.tr

E-mail address: npancaroglu@aku.edu.tr