Asymptotically $\mathscr{I}_2$-Invariant Equivalence of Double Sequences and Some Properties

Erdinç Dündar[*1], **Uğur Ulusu**[†2], **Fatih Nuray**[‡3]

[1,2,3]Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey

Keywords:
Asymptotically equivalence, double sequences, statistical convergence, $\mathscr{I}$-convergence, invariant convergence.
MSC:34C41, 40A35, 40G15

Abstract: In this paper, we give definitions of asymptotically ideal equivalent, asymptotically invariant equivalent and strongly asymptotically invariant equivalent for double sequences. Also, we give some properties and examine the existence relationships among these new type equivalence concepts.

1. Introduction and Background

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1,1,1,...)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$. It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \ \text{uniformly in } n \right\}.$$

Several authors have studied invariant convergent sequences (see, [**? ? ? ? ? ? ? ? ? ? ?**]). The concept of strongly σ-convergence was defined by Mursaleen in [**?**]:

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.

*edundar@aku.edu.tr
†ulusu@aku.edu.tr
‡fnuray@aku.edu.tr

In the case $\sigma(n)=n+1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly σ-convergence was generalized by Savaş [**?**] as below:

$$[V_\sigma]_p=\left\{x=(x_k):\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}|x_{\sigma^k(n)}-L|^p=0,\text{ uniformly in } n\right\},$$

where $0<p<\infty$.

If $p=1$, then $[V_\sigma]_p=[V_\sigma]$. It is known that $[V_\sigma]_p\subset\ell_\infty$.

The idea of statistical convergence was introduced by Fast [**?**] and studied by many authors. The concept of σ-statistically convergent sequence was introduced by Savaş and Nuray in [**?**]. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [**?**] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of the set of natural numbers $\mathbb{N}$. Similar concepts can be seen in [**? ?**].

A family of sets $\mathscr{I}\subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset\in\mathscr{I}$, (ii) For each $A,B\in\mathscr{I}$ we have $A\cup B\in\mathscr{I}$, (iii) For each $A\in\mathscr{I}$ and each $B\subseteq A$ we have $B\in\mathscr{I}$.

An ideal is called non-trivial if $\mathbb{N}\notin\mathscr{I}$ and non-trivial ideal is called admissible if $\{n\}\in\mathscr{I}$ for each $n\in\mathbb{N}$.

Recently, the concepts of σ-uniform density of subset A of the set $\mathbb{N}$ and corresponding $\mathscr{I}_\sigma$-convergence for real number sequences was introduced by Nuray et al. [**?**]. Marouf [**?**] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many other researchers (see, [**? ? ?**]).

Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k}=1$. It is denoted by $x\sim y$.

Convergence and $\mathscr{I}$-convergence of double sequences in a metric space and some properties of this convergence, and similar concepts which are noted following can be seen in [**? ? ? ?**].

A double sequence $x=(x_{kj})$ is said to be bounded if $\sup_{k,j}x_{kj}<\infty$. The set of all bounded double sequences of sets will be denoted by ℓ_∞^2.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i\in N$.

It is evident that a strongly admissible ideal is admissible also.

Let (X,ρ) be a metric space and $\mathscr{I}_2$ be a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. A sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2$-convergent to $L\in X$, if for any $\varepsilon>0$

$$A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},L)\geq\varepsilon\right\}\in\mathscr{I}_2.$$

It is denoted by $\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L$.

Let $A\subseteq\mathbb{N}\times\mathbb{N}$ and

$$s_{mn}:=\min_{k,j}\left|A\cap\left\{\left(\sigma(k),\sigma(j)\right),\left(\sigma^2(k),\sigma^2(j)\right),...,\left(\sigma^m(k),\sigma^n(j)\right)\right\}\right|$$

and

$$S_{mn}:=\max_{k,j}\left|A\cap\left\{\left(\sigma(k),\sigma(j)\right),\left(\sigma^2(k),\sigma^2(j)\right),...,\left(\sigma^m(k),\sigma^n(j)\right)\right\}\right|.$$

If the following limits exists

$$\underline{V_2}(A):=\lim_{m,n\to\infty}\frac{s_{mn}}{mn}\quad\text{and}\quad\overline{V_2}(A):=\lim_{m,n\to\infty}\frac{S_{mn}}{mn},$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A)=\overline{V_2}(A)$, then $V_2(A)=\underline{V_2}(A)=\overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathscr{I}_2^\sigma$ the class of all $A\subseteq\mathbb{N}\times\mathbb{N}$ with $V_2(A)=0$.

Throughout the paper we let $\mathscr{I}_2^\sigma\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

Dündar et al. [**?**] studied the concepts of invariant convergence, strongly invariant convergen, p-strongly invariant convergen and ideal invariant convergence of double sequences.

A double sequence $x=(x_{kj})$ is said to be $\mathscr{I}_2$-invariant convergent or $\mathscr{I}_2^\sigma$-convergent to L if for every $\varepsilon>0$

$$A(\varepsilon)=\left\{(k,j):|x_{kj}-L|\geq\varepsilon\right\}\in\mathscr{I}_2^\sigma,$$

that is, $V_2\big(A(\varepsilon)\big)=0$. In this case, we write $\mathscr{I}_2^\sigma-\lim x=L$ or $x_{kj}\to L(\mathscr{I}_2^\sigma)$.
The set of all $\mathscr{I}_2$-invariant convergent double sequences will be denoted by $\mathfrak{I}_2^\sigma$.
A double sequence $x=(x_{kj})$ is said to be strongly invariant convergent to L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|x_{\sigma^k(s),\sigma^j(t)}-L\right|=0,$$

uniformly in s,t. In this case, we write $x_{kj}\to L\big([V_\sigma^2]\big)$.
A double sequence $x=(x_{kj})$ is said to be p-strongly invariant convergent to L, if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|x_{\sigma^k(s),\sigma^j(t)}-L\right|^p=0,$$

uniformly in s,t, where $0<p<\infty$. In this case, we write $x_{kj}\to L\big([V_\sigma^2]_p\big)$.
The set of all p-strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.
Hazarika [?] introduced the notion of asymptotically $\mathscr{I}$-equivalent sequences and investigated some properties of it. Definitions of P-asymptotically equivalence, asymptotically statistical equivalence and asymptotically $\mathscr{I}_2$-equivalence of double sequences were presented by Hazarika and Kumar [?] as following:
Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be P-asymptotically equivalent if

$$P-\lim_{k,l}\frac{x_{kl}}{y_{kl}}=1,$$

denoted by $x\sim^P y$.
Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon>0$

$$P-\lim_{m,n}\frac{1}{mn}\left|\left\{k\le m,l\le n:\left|\frac{x_{kl}}{y_{kl}}-L\right|\right\}\right|=0,$$

denoted by $x\sim^{\mathscr{S}^L} y$ and simply asymptotically statistical equivalent if $L=1$.
Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be asymptotically $\mathscr{I}_2$-equivalent of multiple L provided that for every $\varepsilon>0$

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\ge\varepsilon\right\}\in\mathscr{I}_2.$$

denoted by $x\sim^{\mathscr{I}_2^L} y$ and simply asymptotically $\mathscr{I}_2$-equivalent if $L=1$.

2. Asymptotically $\mathscr{I}_2^\sigma$-Equivalence

Definition 2.1 Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically invariant equivalent or asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}=L,$$

uniformly in s,t. In this case, we write $x\overset{V_{2(L)}^\sigma}{\sim}y$ and simply σ_2-asymptotically equivalent, if $L=1$.

Definition 2.2 Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically $\mathscr{I}_2^\sigma$-equivalent of multiple L if for every $\varepsilon>0$,

$$A_\varepsilon:=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\ge\varepsilon\right\}\in\mathscr{I}_2^\sigma,$$

i.e., $V_2(A_\varepsilon)=0$. In this case, we write $x\overset{\mathscr{I}_{2(L)}^\sigma}{\sim}y$ and simply asymptotically $\mathscr{I}_2^\sigma$-equivalent, if $L=1$.
The set of all asymptotically $\mathscr{I}_2^\sigma$-equivalent of multiple L sequences will be denoted by $\mathfrak{I}_{2(L)}^\sigma$.

Theorem 2.3 *Suppose that $x=(x_{kl})$ and $y=(y_{kl})$ are bounded double sequences. If x and y are asymptotically $\mathscr{I}_2^\sigma$-equivalent of multiple L, then these sequences are σ_2-asymptotically equivalent of multiple L.*

Proof. Let $m,n,s,t \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Now, we calculate

$$u(m,n,s,t) := \left| \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|.$$

We have

$$u(m,n,s,t) \leq u^{(1)}(m,n,s,t) + u^{(2)}(m,n,s,t),$$

where

$$u^{(1)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \geq \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|$$

and

$$u^{(2)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| < \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|.$$

We get $u^{(2)}(m,n,s,t) < \varepsilon$, for every $s,t = 1,2,\ldots$. The boundedness of $x = (x_{kl})$ and $y = (y_{kl})$ implies that there exists a $M > 0$ such that

$$\left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \leq M,$$

for $k,l = 1,2,...,\ s,t = 1,2,\ldots$. Then, this implies that

$$\begin{aligned} u^{(1)}(m,n,s,t) &\leq \frac{M}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \\ &\leq M \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} = M \frac{S_{mn}}{mn}, \end{aligned}$$

hence x and y are σ_2-asymptotically equivalent to multiple L. □

The converse of Theorem **??** does not hold. For example, $x = (x_{kl})$ and $y = (y_{kl})$ are the double sequences defined by following;

$$x_{kl} := \begin{cases} 2 \quad , & \text{if } k+l \text{ is an even integer,} \\ 0 \quad , & \text{if } k+l \text{ is an odd integer.} \end{cases}$$

$$y_{kl} := 1$$

When $\sigma(m) = m+1$ and $\sigma(n) = n+1$, this sequences are asymptotically σ_2-equivalent but they are not asymptotically $\mathscr{I}_2^{\sigma}$-equivalent.

Definition 2.4 Two nonnegative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be strongly asymptotically invariant equivalent or strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| = 0,$$

uniformly in s,t. In this case, we write $x \overset{[V_{2(L)}^{\sigma}]}{\sim} y$ and simply strongly asymptotically σ_2-equivalent if $L = 1$.

Definition 2.5 Let $0 < p < \infty$. Two nonnegative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be p-strongly asymptotically invariant equivalent or p-strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p = 0,$$

uniformly in s,t. In this case, we write $x \overset{[V_{2(L)}^{\sigma}]_p}{\sim} y$ and simply p-strongly asymptotically σ_2-equivalent if $L = 1$. The set of all p-strongly asymptotically σ_2-equivalent of multiple L sequences will be denoted by $[\mathscr{V}_{2(L)}^{\sigma}]_p$.

Theorem 2.6 *Let $0 < p < \infty$. Then, $x \overset{[\mathscr{V}_{2(L)}^{\sigma}]_p}{\sim} y \Rightarrow x \overset{\mathscr{I}_{2(L)}^{\sigma}}{\sim} y$.*

Proof. Let $x \overset{[\mathscr{V}_{2(L)}^{\sigma}]_p}{\sim} y$ and given $\varepsilon > 0$. Then, for every $s,t \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p &\geq \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\
&\geq \varepsilon^p \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \\
&\geq \varepsilon^p \max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p &\geq \varepsilon^p \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} \\
&= \varepsilon^p \frac{S_{mn}}{mn}
\end{aligned}$$

for every $s,t = 1,2,\ldots$. This implies $\lim_{m,n\to\infty} \frac{S_{mn}}{mn} = 0$ and so $x \overset{\mathscr{I}_{2(L)}^{\sigma}}{\sim} y$.

□

References

[1] P. Das, P. Kostyrko, W. Wilczyński and P. Malik, *$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, 58(5) (2008), 605-620.

[2] E. Dündar and B. Altay, *$\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences*, Acta Mathematica Scientia, 34B(2) (2014), 343-353.

[3] E. Dündar, U. Ulusu and F. Nuray, *On ideal invariant convergence of double sequences and some properties,* Creative Mathematics and Informatics, **27**(2) (2018), (in press).

[4] B. Hazarika, *On asymptotically ideal equivalent sequences*, Journal of the Egyptian Mathematical Society, **23** (2015), 67–72.

[5] B. Hazarika, V. Kumar, *On asymptotically double lacunary statistical equivalent sequences in ideal context*, Journal of Inequlities and Applications, **2013**:543 (2013), 1–15.

[6] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

[7] P. Kostyrko, M. Macaj, T. Šalát and M. Sleziak, *$\mathscr{I}$-Convergence and Extermal $\mathscr{I}$-limits points*, Math. Slovaca, **55** (2005), 443–464.

[8] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathscr{I}$-Convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.

[9] V. Kumar, *On $\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Commun. **12** (2007), 171–181.

[10] M. Marouf, *Asymptotic equivalence and summability*, Int. J. Math. Math. Sci., **16**(4) (1993), 755–762.

[11] M. Mursaleen, *On finite matrices and invariant means*, Indian J. Pure Appl. Math., **10** (1979), 457–460.

[12] M. Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math., **9** (1983), 505–509.

[13] M. Mursaleen and O. H. H. Edely, *On the invariant mean and statistical convergence*, Appl. Math. Lett., **22**(11) (2009), 1700–1704.

[14] F. Nuray, H. Gök and U. Ulusu, *$\mathscr{I}_\sigma$-convergence*, Math. Commun., **16** (2011), 531–538.

[15] F. Nuray and E. Savaş, *Invariant statistical convergence and A-invariant statistical convergence*, Indian J. Pure Appl. Math., **25**(3) (1994), 267–274.

[16] R. F. Patterson, *On asymptotically statistically equivalent sequences*, Demostratio Mathematica, **36**(1) (2003), 149-153.

[17] R. F. Patterson and E. Savaş, *On asymptotically lacunary statistically equivalent sequences*, Thai J. Math., **4**(2) (2006), 267-272.

[18] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.*, **53** (1900), 289-321.

[19] R. A. Raimi, *Invariant means and invariant matrix methods of summability*, Duke Math. J., **30**(1) (1963), 81–94.

[20] E. Savaş, *Some sequence spaces involving invariant means*, Indian J. Math., **31** (1989), 1–8.

[21] E. Savaş, *Strongly σ-convergent sequences*, Bull. Calcutta Math., **81** (1989), 295–300.

[22] E. Savaş, *On $\mathscr{I}$-asymptotically lacunary statistical equivalent sequences*, Adv. Difference Equ., **2013**:111 (2013), 7 pages. doi:10.1186/1687-1847-2013-111

[23] E. Savaş and F. Nuray, *On σ-statistically convergence and lacunary σ-statistically convergence*, Math. Slovaca, **43**(3) (1993), 309–315.

[24] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc., **36** (1972), 104–110.

[25] U. Ulusu, *Asymptotically ideal invariant equivalence*, Creative Mathematics and Informatics, **27**(2) (2018), (in press).