Dependence on Roughness Degree in 2-Normed Spaces

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Abstract

Recently, in summability theory, some mathematicians have studied the concepts of convergence on 2-normed space and rough convergence in normed space. The concept of 2-normed spaces was initially introduced by Gähler in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan (2009) studied statistical convergence and statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık (2008) investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi (2011) studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Statistical convergence of sequences of functions in 2-normed spaces was studied by Yegül and Dündar (2017). Arslan and Dündar (2018a) investigated the concepts of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of functions in 2-normed spaces. Also, Arslan and Dündar (2018b) introduced the concepts of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces.

Phu (2001) first introduced the idea of rough convergence in finite-dimensional normed spaces. In [19], he showed that the set $\mathrm{LIM}^r x_i$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence and investigated some properties. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x_i$ on the roughness degree r. Arslan and Dündar (2018c) defined the concepts of r-convergence and r-Cauchy sequence in 2-normed space and also investigated some properties such as convexity, boundedness and closeness of r-convergence.

In this paper, we examined the dependence of r-limit $\mathrm{LIM}_2^r x_n$ of a fixed sequence (x_n) on varying parameter r in 2-normed space.

Key Words: 2-normed space, Rough convergence, Rough limit point and Roughness degree.

Özet

Son zamanlarda, toplanabilme teorisinde, bazı matematikçiler 2-normlu uzaylarda yakınsaklık ve normlu uzaylarda rough yakınsaklık kavramlarını çalışmaktadır. 2-normlu uzay kavramı ilk olarak 1960 lı yıllarda Gähler tarafından tanımlandı. O zamandan bu yana, bu kavram bir çok matematikçi tarafından detaylı bir şekilde çalışılmakta ve incelenmektedir. Gürdal ve Pehlivan (2009) 2-normlu uzaylarda istatistiksel yakınsaklık ve istatistiksel Cauchy dizisi üzerine çalıştılar ve istatistiksel yakınsaklığın bazı özelliklerini incelediler. Gürdal ve Açık (2008) 2-normlu uzaylarda $\mathcal{I}$-Cauchy dizisi ve $\mathcal{I}^*$-Cauchy dizisini incelediler. Sarabadan ve Talebi (2011) 2- normlu uzaylarda fonksiyon dizilerinin istatistiksel ve ideal yakınsaklığı üzerine çalıştılar. Arslan ve Dündar (2018) 2-normlu uzaylarda $\mathcal{I}$-yakınsaklık ve $\mathcal{I}^*$-yakınsaklık kavramı ile $\mathcal{I}$-Cauchy dizisi ve $\mathcal{I}^*$-Cauchy dizisi kavramlarını araştırdılar.

Rough yakınsaklık kavramı ilk olarak Phu (2001) tarafından sonlu boyutlu normlu uzaylar için tanımlandı. Bu çalışmada Phu, $\mathrm{LIM}^r x_i$ kümesinin sınırlı, kapalı ve konveks olduğunu gösterdi. Ayrıca, Phu bu çalışmada rough Cauchy dizisi kavramını tanımladı ve bazı özelliklerini inceledi. Aynı zamanda Phu bu çalışmada, rough yakınsaklık ile diğer yakınsaklık türleri arasındaki ilişkiyi ve $\mathrm{LIM}^r x_i$ kümesinin rough lık derecesi olan r ye bağlılığını ayrıntılı bir şekilde inceledi.

Bu çalışmada, 2-normlu uzaylarda sabit bir (x_n) dizisinin r- limiti olan $\mathrm{LIM}_2^r x_n$ kümesinin değişken r parametresine bağlılığını inceledik.

Anahtar Kelimeler: 2-normlu uzay, Rough yakınsaklık, Rough limit noktası ve Roughlık derecesi.

Introduction and Definitions

Let X be a real vector space of dimension $d,$ where $2 \leq d < \infty$.
A 2-norm on X is a function $\|\cdot,\cdot\|\colon X \times X \to R$ which satisfies the following statements :

(i) $\| x, y \| = 0$ *if and only if x and y are linearly dependent.*
(ii) $\| x, y \| = \| y, x \|$.
(iii) $\| \alpha x, y \| = |\alpha| \| x, y \|, \alpha \in R$.
(iv) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$.

As an example of a 2-normed space we may take $X = R^2$ being equipped with the 2-norm $\|x,y\|$:= the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \;\; x = (x_1, x_2), y = (y_1, y_2) \in R^2.$$

In this study, we suppose X to be a 2-normed space having dimension d where $2 \leq d < \infty$. The pair $(X,\|\cdot,\cdot\|)$ is then called 2-normed space.

A sequence (x_n) in 2-normes space $(X,\|\cdot,\cdot\|)$ is said to be convergent to L in X if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0,$$

for every $y \in X$. In such a case, we write

$$\lim_{n\to\infty} x_n = L$$

and call L the limit of (x_n).

Example1.1. Let $x = (x_n) = \left(\frac{n}{n+1}, \frac{1}{n}\right)$, $L = (1,0)$ *and* $z = (z_1, z_2)$. It is clear that (x_n) convergent to $L = (1,0)$ in 2-normed space.

Let r be a nonnegative real number and R^n denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_i) \subset R^n$.

The sequence $x = (x_i)$ is said to be r-convergent to x_* , denote by $x_i \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0, \exists i_\varepsilon \in N\colon i \geq i_\varepsilon \implies \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x_i \coloneqq \{x_* \in R^n\colon x_i \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathrm{LIM}^r x_i \neq \emptyset$. In this case r is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence.

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics

i. Direct proof method,
ii. Inverse situation proof method,
iii. Method of non-finding (contradiction),
iv. Induction method

methods were used as needed.

Main Results

Definition 1.1. Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a nonnegative real number. (x_n) is said to be rough convergent (r-convergent) to L denoted by $x_n \overset{\|.,.\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0\ \exists n_\varepsilon \in N: n \geq n_\varepsilon \implies \|x_n - L\| < r + \varepsilon \qquad (2.1)$$

or equivalently, if,

$$\limsup \|x_n - L, z\| \leq r,$$

for every $z \in X$.

If (2.1) holds L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\mathrm{LIM}_2^r x_n := \{L \in X: x_n \overset{\|.,.\|}{\longrightarrow}_r L\}.$$

The sequence (x_n) is said to be rough convergent if

$$\mathrm{LIM}_2^r x_n \neq \emptyset.$$

Now, we investigate the dependence of r-limit $\mathrm{LIM}_2^r x_n$ of a fixed sequence (x_n) on a varying parameter r.

It follows from definition

$$\mathrm{LIM}_2^{r_1} x_n \subset \mathrm{LIM}_2^{r_2} x_n, \quad if \quad r_1 < r_2. \qquad (2.2)$$

This monotonicity is included in the following :

Theorem 1.1. Let $(X, \|.,.\|)$ be a 2-normed space and suppose $r \geq 0$ and $\rho > 0$. Then,

a. $\mathrm{LIM}_2^r x_n + \bar{B}_r(0) \subseteq \mathrm{LIM}_2^{r+\rho} x_n$.

b. $\bar{B}_\rho(y) \subseteq \mathrm{LIM}_2^r x_n$ implies $y \in \mathrm{LIM}_2^{r-\rho} x_n$.

Now, define

$$\bar{r} := \inf\{r > 0 : \mathrm{LIM}_2^r x_n \neq \emptyset\}.$$

By the monotonicity given in (2.2) , we have

$$\mathrm{LIM}_2^r x_n = \begin{cases} = \emptyset, & r < \bar{r} \\ \neq \emptyset, & r > \bar{r}. \end{cases}$$

Morover, by Theorem 2.1, for all $r > \bar{r}$ and $\rho \in (0, r - \bar{r})$, $\mathrm{LIM}_2^r x_n$ always contains some ball with radius ρ, that means at least

$$int(\mathrm{LIM}_2^r x_n) \neq \emptyset \quad for\, r > \bar{r}.$$

Therefore,

$$int(\mathrm{LIM}_2^r x_n) = \emptyset \text{ implies } r \leq \bar{r} \ \ and\, \mathrm{LIM}_2^{r'} x_n = \emptyset \ \ for\ \ r' \in [0, r).$$

Theorem 1.2. Let $(X, \|.,.\|)$ be a 2-normed space.

a. $r = \bar{r}$ if and only if $\mathrm{LIM}_2^r x_n \neq \emptyset$ and $int(\mathrm{LIM}_2^r x_n) = \emptyset$.

b. $If\ (X, \|.,.\|)$ is a finite- dimensional strictly convex space then $r = \bar{r}$ if and only if $\mathrm{LIM}_2^r x_n$ is a singleton.

Theorem 1.3. Let $(X, \|.,.\|)$ be a 2-normed space. The following holds :

$$cl\left(\bigcup_{\mathbf{0}\leq \boldsymbol{r'}<r} \mathrm{LIM}_2^{r'} x_n\right) \subseteq \mathrm{LIM}_2^{r} x_n = \bigcap_{r'>r} \mathrm{LIM}_2^{r'} x_n.$$

If $r \neq \bar{r}$ then

$$cl\left(\bigcup_{\mathbf{0}\leq \boldsymbol{r'}<r} \mathrm{LIM}_2^{r'} x_n\right) = \mathrm{LIM}_2^{r} x_n.$$

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