On Some Propertıes of Rough Convergence In 2-Normed Spaces

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Abstract

The concepts of convergence on 2-normed space and rough convergence in normed spaces are important in summability theory. The concept of 2-normed spaces was initially introduced by Gähler in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan (2009) studied statistical convergence and statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık (2008) investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi (2011) studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Yegül and Dündar (2017) investigated statistical convergence of sequences of functions in 2-normed spaces. Arslan and Dündar (2018a, 2018b) investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces.

The idea of rough convergence was first introduced by Phu (2001) in finite-dimensional normed spaces. In [19], he showed that the set $\text{LIM}^r x_i$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence and investigated some properties. He also investigated the relations between rough convergence and other convergence types and the dependence of $\text{LIM}^r x_i$ on the roughness degree r. Arslan and Dündar (2018c) defined r-convergence and r-Cauchy sequence in 2-normed space and also investigated some properties of r-convergence.

In this study, we investigated relationships between rough convergence and classical convergence and studied some properties about the notion of rough convergence, the set of rough limit points and rough cluster points of a sequence in 2-normed space. Also, we examined the dependence of r-limit $\text{LIM}_2^r x_n$ of a fixed sequence (x_n) on varying parameter r in 2-normed space.

Key Words: Rough convergence, 2-normed space, rough limit point and rough cluster point.

Özet

2-normlu uzaylarda yakınsaklık ve normlu uzaylarda rough yakınsaklık kavramları toplanabilme teorisinde önemlidir. 2-normlu uzay kavramı ilk olarak 1960 lı yıllarda Gähler tarafından tanımlandı. O zamandan bu yana, bu kavram bir çok matematikçi tarafından detaylı bir şekilde çalışılmakta ve incelenmektedir. Gürdal ve Pehlivan (2009) 2-normlu uzaylarda istatistiksel yakınsaklık ve istatistiksel Cauchy dizisi üzerine çalıştılar ve istatistiksel yakınsaklığın bazı özelliklerini incelediler. Gürdal ve Açık (2008) 2-normlu uzaylarda $\mathcal{I}$-Cauchy dizisi ve $\mathcal{I}^*$-Cauchy dizisini incelediler. Sarabadan ve Talebi (2011) 2-normlu uzaylarda fonksiyon dizilerin istatistiksel ve ideal yakınsaklığı çalıştılar. Yegül ve Dündar (2017) 2-normlu uzaylarda fonksiyon dizilerinin istatistiksel yakınsaklığını araştırdılar. Arslan ve Dündar (2018a, 2018b) 2-normlu uzaylarda $\mathcal{I}$-yakınsaklık ve $\mathcal{I}^*$-yakınsaklık kavramı ile $\mathcal{I}$-Cauchy dizisi ve $\mathcal{I}^*$-Cauchy dizisi kavramlarını araştırdılar.

Rough yakınsaklık kavramı ilk olarak Phu (2001) tarafından sonlu boyutlu normlu uzaylar için tanımlandı. Bu çalışmada Phu, $\text{LIM}^r x_i$ kümesinin sınırlı, kapalı ve konveks olduğunu gösterdi. Ayrıca, Phu bu çalışmada rough Cauchy dizisi kavramını tanımladı ve bazı özelliklerini inceledi. Aynı zamanda Phu bu çalışmada, rough yakınsaklık ile diğer yakınsaklık türleri arasındaki ilişkiyi ve $\text{LIM}^r x_i$ kümesinin rough lık derecesi olan r ye bağlılığını ayrıntılı bir şekilde inceledi. Arslan ve Dündar (2018c) 2-normlu uzaylarda r-yakınsaklık ve r-Cauchy dizisi kavramlarını tanımladılar ve r-yakınsaklığın bazı özelliklerini incelediler.

Bu çalışmada, 2-normlu uzaylarda rough yakınsaklık ile klasik anlamdaki yakınsaklık arasındaki ilişkiyi ve rough yakınsaklık gösteriminin bazı özelliklerini, rough limit noktaları kümesini ve rough yığılma noktaları kümesini araştırdık. Aynı zamanda, 2-normlu uzaylarda sabit bir (x_n) dizisinin r-limiti olan $\text{LIM}_2^r x_n$ kümesinin değişken r parametresine bağlılığını inceledik.

Anahtar Kelimeler: Rough yakınsaklık, 2-normlu uzay, rough limit noktası ve rough yığılma noktası.

Introduction and Definitions

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\|: X \times X \to R$
Which satisfies the following statements :

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent.
(ii) $\| x, y \| = \| y, x \|$.
(iii) $\| \alpha x, y \| = |\alpha| \, \| x, y \|, \alpha \in R$.
(iv) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$.

As an example of a 2-normed space we may take $X = R^2$ being equipped with the 2-norm ‖x,y‖ := the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \ x = (x_1, x_2), y = (y_1, y_2) \epsilon \ R^2.$$

In this study, we suppose X to be a 2-normes space having dimension d where $2 \leq d < \infty$. The pair $(X,\|\cdot,\cdot\|)$ is then called 2-normed space.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be convergent to L in X if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0,$$

for every y ∈ X. In such a case, we write

$$\lim_{n\to\infty} x_n = L$$

and call L the limit of (x_n).

Example1.1. Let $x = (x_n) = \left(\frac{n}{n+1}, \frac{1}{n}\right)$, $L = (1,0)$ and $z = (z_1, z_2)$. It is clear that (x_n) convergent to $L = (1,0)$ in 2-normed space.

Let r be a nonnegative real number an R^n denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_i) \subset R^n$.

The sequence $x = (x_i)$ is said to be r-convergent to x_*, denote by $x_i \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0, \exists i_\varepsilon \in N: i \geq i_\varepsilon \implies \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\text{LIM}^r x_i := \{x_* \in R^n: x_i \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be r-convergent if $\text{LIM}^r x_i \neq \emptyset$. In this case r is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence.

Lemma 1.1. ([3], Theorem 2.2). Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. The sequence (x_n) is bounded if and only if there exists an $r > 0$ such that $\text{LIM}_2^r x_n \neq \emptyset$. For all $r > 0$, a bounded sequence (x_n) is always contains a subsequence (x_{n_k}) with

$$\text{LIM}_2^{x_{n_k}, r} x_{n_k} \neq \emptyset.$$

Lemma 1.2. ([3], Theorem 2.3). Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. For all $r \geq 0$, the r-limit set $\text{LIM}_2^r x_n$ of an arbitrary sequence (x_n) is closed.

Lemma 1.3. ([3], Theorem 2.4). Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. If $y_0 \in \mathrm{LIM}_2^{r_0}\, x_n$ and $y_1 \in \mathrm{LIM}_2^{r_1}\, x_n$, then ,

$$y_\alpha \coloneqq (1-\alpha)y_0 + \alpha y_1 \epsilon\, y_0 \in \mathrm{LIM}_2^{(1-\alpha)r_0+\alpha r_1}\, x_n,$$

$for\ \alpha \in [0,1]$.

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics

i. Direct proof method,
ii. Inverse situation proof method,
iii. Method of non-finding (contradiction),
iv. Induction method

methods were used as needed.

Main Result

Definition 2.1. Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a nonnegative real number. (x_n) is said to be rough convergent (r-convergent) to L denoted by $x_n \overset{\|.,.\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0 \; \exists n_\varepsilon \in N : n \geq n_\varepsilon \Longrightarrow \|x_n - L\| < r + \varepsilon \qquad (2.1)$$

or equivalently, if,

$$\limsup \|x_n - L, z\| \leq r,$$

for every $z \in X$.

If (2.1) holds, L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\mathrm{LIM}_2^r x_n := \{L \in X : x_n \overset{\|.,.\|}{\longrightarrow}_r L\}.$$

The sequence (x_n) is said to be rough convergent if $\mathrm{LIM}_2^r x_n \neq \emptyset$.

Example 1.1. The sequence $x = (x_n) = ((-1)^n, (-1)^n)$ is not convergent in 2-normed space $(X, \|.,.\|)$ but it is rough convergent to $L = (0,0)$, for every $z \in X$. It is clear that

$$\mathrm{LIM}_2^r x_n = \begin{cases} \emptyset, & if\ r < 1 \\ ((-r,-r),(r,r)), & otherwise. \end{cases}$$

Theorem 1.1. Let $(X, \|.,.\|)$ be a 2-normed space and $x = (x_n)$ be a sequence in X, $r_1 \geq 0$ and $r_2 > 0$. $x = (x_n)$ is $(r_1 + r_2)$- convergent to L in X if and only if there exits a sequence $(y_n) \in X$ such that for every $z \in X$

$$y_n \overset{\|.,.\|}{\longrightarrow}_{r_1} L \text{ and } \|x_n - y_n, z\| \leq r_2, \quad n = 1,2,\ldots \ .$$

Theorem 1.2. Let $(X, \|.,.\|)$ be a 2-normed space and (x_n) be a sequence in X. (x_n) converges to L if and only if

$$\mathrm{LIM}_2^r x_n = \bar{B}_r(L)$$

where

$$\bar{B}_r(L) := \{x_1 \in X : \|x_n - L, z\| \leq r\},$$

for every $z \in X$.

Theorem 1.3. Let $(X, \|.,.\|)$ be a 2-normed space. Then,

a. If c is a cluster point of the sequence (x_n) then,

$$\mathrm{LIM}_2^r x_n = \bar{B}_r(c).$$

b. Let C be the set of all cluster points of $(x_n) \subset X$. Then,

$$\mathrm{LIM}_2^r x_n = \bigcap_{c \in C} \bar{B}_r(c) = \{L \in X : C \subseteq \bar{B}_r(L)\}.$$

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