Asymptotically $\mathcal{I}_2^\sigma$-Equivalence of Double Sequences of Sets Defined by Modulus Functions

Erdinç Dündar, Afyon Kocatepe University, Turkey, edundar@aku.edu.tr

Nimet Pancaroğlu Akın, Afyon Kocatepe University, Turkey, npancaroglu@aku.edu.tr

Öz

Fast (1951) ve Schoenberg (1959) tarafından istatistiksel yakınsaklık kavramı tanımlandı. Bu kavram ile ilgili birçok yazar tarafından çeşitli çalışmalar yapıldı. Mursaleen ve Edely (2003) bu kavramı çift küme dizilerine taşımıştur. İstatistiksel yakınsaklığın bir genelleştirmesi olan $\mathcal{I}$-yakınsaklık Kostyrko vd. (2000) tarafından tanımlanmış olup, bu kavram $\mathbb{N}$ doğal sayılar kümesinin alt kümelerinin sınıfı olan $\mathcal{I}$ idealinin yapısına bağlıdır. $\mathcal{I}_2$- yakınsaklık kavramı ve bu kavramın bazı özellikleri Das vd. (2008) tarafından incelendi. Nuray ve Rhoades (2012) küme dizileri için istatistiksel yakınsaklık kavramını tanımlayıp bu kavramla ilgili bazı özellikleri ve teoremleri inceledi. Küme dizilerinin Wijsman $\mathcal{I}$- yakınsaklığı Kişi ve Nuray (2013) tarafından tanımlandı.

Bazı yazarlar [Mursaleen (1983), Pancaroğlu ve Nuray (2013, 2014), Raimi (1963), Savaş ve Nuray (1993)] invaryant yakınsak diziler ile ilgili bazı çalışmalar yaptı. Tortop ve Dündar (2018) çift küme dizilerinde $\mathcal{I}_2$-invariant yakınsaklık ile ilgili bir çalışma yaptı. Akın tarafından çift küme dizilerinin Wijsman lacunary $\mathcal{I}_2$-invariant yakınsaklığı ile ilgili bir çalışma yapıldı. Marouf (1993) asimptotik denklik ve asimptotik regüler matris kavramlarını tanımladı. Modülüs fonksiyonu ilk defa Nakano (1953) tarafından tanımlandı. Maddox (1986), Pehlivan (1995) ve birçok yazar tarafından f modülüs fonksiyonu kullanılarak bazı yeni kavramları ve sonuç teoremlerini içeren çalışmalar yapıldı. Modülüs fonsiyonunu kullanılarak lacunary ideal denk diziler ile ilgili Kumar ve Sharma (2012) tarafından bir çalışma yapıldı. Akın and Dündar (2018) and Akın vd. (2018) tarafından küme dizilerinin f-asimptotik $\mathcal{I}_\sigma$ ve $\mathcal{I}_{\sigma\theta}$-istatistiksel denkliği kıavramlarının tanımları yapıldı.

Bu çalışmada çift küme dizileri için kuvvetli asimptotik$\mathcal{I}_2^\sigma$- denklik, f-asymptotik $\mathcal{I}_2^\sigma$- denklik, kuvvetli$f$-asymptotik $\mathcal{I}_2^\sigma$-denklik kavramları tanımlandı. Daha sonra bu kavramların özellikleri ve aralarındaki ilişkiler incelendi.

Anahtar Kelimeler: Asimptotik Denklik; $\mathcal{I}_2$-Yakınsaklık; Invariant Yakınsaklık; Wijsman Yakınsaklık; Modülüs Fonksiyonu.

Abstract

Fast (1951) and Schoenberg (1959), independently, introduced the concept of statistical convergence and many authors studied this concept. Mursaleen and Edely (2003) extended this concept to the double sequences. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. (2000) as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$. The idea of $\mathcal{I}_2$-convergence and some properties of this convergence were studied by Das et al. (2008). Nuray and Rhoades (2012) defined the idea of statistical convergence of set sequence and investigated some theorems about this notion and important properties. Kişi and Nuray (2013) defined Wijsman $\mathcal{I}$-convergence of sequence of sets.

Several authors have studied invariant convergent sequences [Mursaleen (1983), Pancaroğlu and Nuray (2013, 2014), Raimi (1963), Savaş and Nuray (1993)]. Tortop and Dündar (2018) introduced $\mathcal{I}_2$-invariant convergence of double set sequences. Akın studied Wijsman lacunary $\mathcal{I}_2$-invariant convergence of double sequences of sets. Marouf (1993) peresented definitions for asymptotically equivalent and asymptotic regular matrices. Modulus function was introduced by Nakano (1953). Maddox (1986), Pehlivan (1995) and many authors used a modulus function f to new some new concepts and inclusion theorems. Kumar and Sharma (2012) studied lacunary equivalent sequences by ideals and modulus function. Akın and Dündar (2018) and Akın et al. (2018) give definitions of f-asymptotically $\mathcal{I}_\sigma$ and $\mathcal{I}_{\sigma\theta}$-statistical equivalence of set sequences.

In this study, first, we present the concepts of strongly asymptotically $\mathcal{I}_2^\sigma$- equivalence, f-asymptotically $\mathcal{I}_2^\sigma$- equivalence, strongly f-asymptotically $\mathcal{I}_2^\sigma$- equivalence for double sequences of sets. Then, we investigated some properties and relationships among this new concepts.

Key Words: Asymptotic Equivalence; $\mathcal{I}_2$-Convergence; Invariant Convergence; Wijsman Convergence; Modulus Function.

Introduction and Definitions

Asymptotically equivalent and some properties of equivalence are studied by several authors [see, Kişi et al. (2015), Pancaroğlu et al. (2013), Patterson (2003), Savaş (2013), Ulusu and Nuray (2013)]. Ulusu and Gülle introduced the concept of asymptotically $\mathcal{I}_\sigma$-equivalence of sequences of sets. Recently, Dündar et al. studied on asymptotically ideal invariant equivalence of double sequences.

Several authors define some new concepts and give inclusion theorems using a modulus function f [see, Khan and Khan (2013), Kılınç and Solak (2014), Maddox (1986), Nakano (1953), Pehlivan and Fisher(1995)]. Kumar and Sharma (2012) studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f. Kişi et al. (2015) introduced f-asymptotically $\mathcal{I}_\theta$-equivalent set sequences. Akın and Dündar (2018) and Akın et al. (2018) give definitions of f-asymptotically $\mathcal{I}_\sigma$ and $\mathcal{I}_{\sigma\theta}$-statistical equivalence of set sequences.

Now, we recall the basic concepts and some definitions and notations (see, [Baronti and Papini (1986), Beer (1985, 1994), Das et al. (2008), Dündar et al. (2016, 2017), Fast (1951), Kostyrko et al. (2000), Lorentz (1948), Marouf (1993), Mursaleen (1983), Nakano (1953), Nuray et al. (2011, 2016), Pancaroğlu and Nuray (2014), Akın and Dündar (2018), Pehlivan and Fisher (1995), Raimi (1963), Tortop and Dündar, Ulusu and Dündar (2014) and Wijsman (1964, 1966)]).

Let $u = (u_k)$ and $v = (v_k)$ be two non-negative sequences. If $\lim_k \frac{u_k}{v_k} = 1$, then they are said to be asymptotically equivalent (denoted by $u \sim v$).

Let (Y, ρ) be a metric space, $y \in Y$ and E be any non-empty subset of Y, we define the distance from y to E by

$$d(y, E) = \inf_{e \in E} \rho(y, e).$$

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

1. $\phi(u) \geq 0$, when the sequence $u = (u_j)$ has $u_j \geq 0$ for all j,
2. $\phi(i) = 1$, where $i = (1,1,1\ldots)$,
3. $\phi(\mathrm{u}_{\sigma(j)}) = \phi(u)$, for all $u \in \ell_\infty$.

The mapping ϕ is supposed to be one-to-one and such that $\sigma^m(j) \neq j$ for all positive integers j and m, where $\sigma^m(j)$ denotes the mth iterate of the mapping σ at j. Hence, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(u) = \lim u$ for all $u \in c$. If σ is a translation mapping that is $\sigma(j) = j + 1$, the σ mean is often called a Banach limit.

Let (Y, ρ) be a metric space and E, F, E_i and F_i $(i = 1,2,\ldots)$ be non-empty closed subsets of Y.

Let $L \in \mathbb{R}$. Then, we define $d(y; E_i, F_i)$ as follows:

$$d(y; E_i, F_i) = \begin{cases} \dfrac{d(y, E_i)}{d(y, F_i)}, & y \notin E_i \cup F_i, \\ L, & y \in E_i \cup F_i. \end{cases}$$

Let E_i, $F_i \subseteq Y$. If for each $y \in Y$,

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} \left| d\left(y; E_{\sigma^i(m)}, F_{\sigma^i(m)}\right) - L \right| = 0,$$

uniformly in m, then, the sequences $\{E_i\}$ and $\{F_i\}$ are strongly asymptotically invariant equivalent of multiple L, (denoted by $E_i \overset{[WV]_\sigma^L}{\sim} F_i$) and if $L = 1$, simply strongly asymptotically invariant equivalent.

$\mathcal{I} \subseteq 2^{\mathbb{N}}$ which is a family of subsets of $\mathbb{N}$ is called an ideal, if the followings hold:

(i) $\emptyset \in \mathcal{I}$,

(ii) For each $E, F \in \mathcal{I}$, $E \cup F \in \mathcal{I}$,

(iii) For each $E \in \mathcal{I}$ and each $F \subseteq E$, we have $F \in \mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{I}$. Also, for non-trivial ideal and for each $n \in \mathbb{N}$ if $\{n\} \in \mathcal{I}$, then $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is admissible ideal. After that, we consider that $\mathcal{I}$ is an admissible ideal.

Let $K \subseteq \mathbb{N}$ and

$$s_m = \min_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|$$

and

$$S_m = \max_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|.$$

If the limits $\underline{V}(K) = \lim_{m\to\infty} \frac{s_m}{m}$ and $\overline{V}(K) = \lim_{m\to\infty} \frac{S_m}{m}$ exists then, they are called a lower σ-uniform density and an upper σ-uniform density of the set K, respectively. If $\underline{V}(K) = \overline{V}(K)$, then $V(K) = \underline{V}(K) = \overline{V}(K)$ is called the σ-uniform density of K.

Denote by $\mathcal{I}_\sigma$ the class of all $K \subseteq \mathbb{N}$ with $V(K) = 0$. It is clearly that $\mathcal{I}_\sigma$ is admissible ideal.

If for every $\gamma > 0$, $A_\gamma = \{i: |x_i - L| \geq \gamma\}$ belongs to $\mathcal{I}_\sigma$, i.e., $V(A_\gamma) = 0$ then, the sequence $u = (u_i)$ is said to be $\mathcal{I}_\sigma$-convergent to L. It is denoted by $\mathcal{I}_\sigma - \lim u_i = L$.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$A_{\gamma,y}^{\sim} = \{i: |d(y; E_i, F_i) - L| \geq \gamma\}$$

belongs to $\mathcal{I}_\sigma$, that is, $V(A_{\gamma,y}^{\sim}) = 0$ then, the sequences $\{E_i\}$ and $\{F_i\}$ are asymptotically $\mathcal{I}$-invariant equivalent or asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L. In this instance, we write $E_i \overset{W_{\mathcal{I}_\sigma}^L}{\sim} F_i$ and if $L = 1$, simply asymptotically $\mathcal{I}$-invariant equivalent.

If following conditions hold for the function $f: [0, \infty) \to [0, \infty)$, then it is called a modulus function:

1. $f(u) = 0$ if and if only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$,
3. f is nondecreasing,
4. f is continuous from the right at 0.

This after, we let f as a modulus function.

The modulus function f may be unbounded (for example $f(u) = u^q$, $0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{i=1}^{n} |d(y; E_i, F_i) - L| \geq \gamma\right\} \in \mathcal{I}_\sigma,$$

then, $\{E_i\}$ and $\{F_i\}$ are strongly asymptotically $\mathcal{I}$-invariant equivalent of multiple L (denoted by $E_i \overset{[W_{\mathcal{I}_\sigma}^L]}{\sim} F_i$) and if $L = 1$, simply strongly asymptotically $\mathcal{I}_\sigma$-equivalent.

If for every $\gamma > 0$ and for each $y \in Y$,

$$\{i \in \mathbb{N}: f(|d(y; E_i, F_i) - L|) \geq \gamma\} \in \mathcal{I}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L denoted by $E_i \overset{W_{\mathcal{I}_\sigma}^L(f)}{\sim} F_i$ and if $L = 1$ simply f-asymptotically $\mathcal{I}$-invariant equivalent.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{i=1}^{n} f(|d(y; E_i, F_i) - L|) \geq \gamma\right\} \in \mathcal{I}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be strongly f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L denoted by $E_i \overset{[W^L_{\mathcal{I}_\sigma}(f)]}{\sim} F_i$ and if $L = 1$, simply strongly f-asymptotically $\mathcal{I}$-invariant equivalent.

Let $\mathcal{I}_2$ be a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$. It is called strongly admissible ideal if $\{k\} \times \mathbb{N}$ and $\mathbb{N} \times \{k\}$ belong to $\mathcal{I}_2$ for each $k \in N$. This after, we let $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

If we let a ideal as a strongly admissible ideal then, it is clear that it is admissible also.

Let

$$\mathcal{I}_2^0 = \{E \subset \mathbb{N} \times \mathbb{N}: (\exists\ i(E) \in \mathbb{N})(r, s \geq i(E) \Rightarrow (r, s) \not\in E)\}.$$

It is clear that $\mathcal{I}_2^0$ is a strongly admissible ideal. Also, it is evidently $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (Y, ρ) be a metric space and $y = (y_{ij})$ be a sequence in Y. If for any $\gamma > 0$,

$$A(\gamma) = \{(i, j) \in \mathbb{N} \times \mathbb{N}: \rho(y_{ij}, L) \geq \gamma\} \in \mathcal{I}_2$$

then, it is said to be $\mathcal{I}_2$-convergent to L. In this instance, y is $\mathcal{I}_2$-convergent and we write $\mathcal{I}_2 - \lim_{i,j\to\infty} y_{ij} = L$.

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mk}: \min_{i,j} |\mathrm{E} \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|$$

and

$$S_{mk}: \max_{i,j} |E \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|.$$

If the limits $\underline{V_2}(E) := \lim_{m,k\to\infty} \frac{s_{mk}}{mk}$, $\overline{V_2}(E) := \lim_{m,k\to\infty} \frac{S_{mk}}{mk}$ exists then $\underline{V_2}(E)$ is called a lower and $\overline{V_2}(E)$ is called an upper σ-uniform density of the set E, respectively. If $\underline{V_2}(E) = \overline{V_2}(E)$ holds then, $V_2(E) = \underline{V_2}(E) = \overline{V_2}(E)$ is called the σ-uniform density of E.

Denote by $\mathcal{I}_2^\sigma$ the class of all $E \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(E) = 0$.

This after, let (Y, ρ) be a separable metric space and E_{ij}, F_{ij}, E, F be any nonempty closed subsets of Y.

If for each $y \in Y$,

$$\lim_{m,k\to\infty} \frac{1}{mk} \sum_{i,j=1,1}^{m,k} d\left(y, E_{\sigma^i(s), \sigma^j(t)}\right) = d(y, E),$$

uniformly in *s*,*t* then, the double sequence $\{E_{ij}\}$ is said to be invariant convergent to E in Y.

If for every $\gamma > 0$,

$$A(\gamma, y) = \{(i, j): |d(y, E_{ij}) - d(y, E)| \geq \gamma\} \in \mathcal{I}_2^\sigma$$

that is, $V_2(A(\gamma, y)) = 0$, then, the double sequence $\{E_{ij}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_{W_2}^\sigma$-convergent to E, In this instance, we write $E_{ij} \to E(\mathcal{I}_{W_2}^\sigma)$ and by $\mathcal{I}_{W_2}^\sigma$ we will denote the set of all Wijsman $\mathcal{I}_2^\sigma$-convergent double sequences of sets.

For non-empty closed subsets E_{ij}, F_{ij} of Y define $d(y; E_{ij}, F_{ij})$ as follows:

$$d(y; E_{ij}, F_{ij}) = \begin{cases} \dfrac{d(y, E_{ij})}{d(y, F_{ij})}, & y \not\in E_{ij} \cup F_{ij} \\ L, & y \in E_{ij} \cup F_{ij}. \end{cases}$$

Lemma 1. *[*Pehlivan and Fisher, 1995*] Let* $0 < \gamma < 1$. *Thus, for each* $u \geq \gamma,\ f(u) \leq 2f(1)\gamma^{-1}u$.

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics,

i.Direct proof method,

ii.Reverse proof method

iii.Contrapositive method,

iv.Induction method

methods were used as needed.

Main Results

Definition 2.1 If for every $\gamma > 0$ and each $y \in Y$,

$$\left\{(m,k): \in \mathbb{N} \times \mathbb{N}: \frac{1}{mk} \sum_{i,j=1,1}^{m,k} |d(y; E_{ij}, F_{ij}) - L| \geq \gamma \right\} \in \mathcal{I}_2^{\sigma}$$

then, double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be strongly asymptotically $\mathcal{I}_2$-invariant equivalent of multiple L denoted by

$$E_{ij} \overset{[W_{\mathcal{I}_2^{\sigma}}^{L}]}{\sim} F_{ij}$$

and if $L = 1$, simply strongly asymptotically $\mathcal{I}_2^{\sigma}$-equivalent.

Definition 2.2 If for every $\gamma > 0$ and each $y \in Y$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: f(|d(y; E_{ij}, F_{ij}) - L|) \geq \gamma\} \in \mathcal{I}_2^{\sigma}$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be f-asymptotically $\mathcal{I}_2$-invariant equivalent of multiple L denoted by

$$E_{ij} \overset{W_{\mathcal{I}_2^{\sigma}}^{L}(f)}{\sim} F_{ij}$$

and if $L = 1$, simply f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalent.

Definition 2.3 If for every $\gamma > 0$ and each $y \in Y$,

$$\left\{(m,k) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mk} \sum_{i,j=1,1}^{mk} f(|d(y; E_{ij}, F_{ij}) - L|) \geq \gamma \right\} \in \mathcal{I}_2^{\sigma}$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalent of multiple L denoted by

$$E_{ij} \overset{[W_{\mathcal{I}_2^{\sigma}}^{L}(f)]}{\sim} F_{ij}$$

and if $L = 1$, simply strongly f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalent.

Theorem 2.1 For each $y \in Y$, we have

$$E_{ij} \overset{[W_{\mathcal{I}_2^{\sigma}}^{L}]}{\sim} F_{ij} \Rightarrow E_{ij} \overset{[W_{\mathcal{I}_2^{\sigma}}^{L}(f)]}{\sim} F_{ij}.$$

Theorem 2.2 Let $z \in Y$. If $\lim_{z\to\infty} \frac{f(z)}{z} = \alpha > 0$, then

$$E_{ij} \overset{[W_{\mathcal{J}_2\sigma}^{L}]}{\sim} F_{ij} \Leftrightarrow E_{ij} \overset{[W_{\mathcal{J}_2\sigma}^{L}(f)]}{\sim} F_{ij}.$$

References

Baronti M., and Papini P.,(1986). *Convergence of sequences of sets*. In: Methods of functional analysis in approximation theory (pp. 133-155), ISNM 76, Birkhäuser, Basel.

Beer G., (1985). *On convergence of closed sets in a metric space and distance functions*. Bulletin of the Australian Mathematical Society, **31**, 421-432.

Beer G., (1994). *Wijsman convergence*: A survey. Set-Valued Analysis, **2** , 77-94.

Das, P., Kostyrko, P., Wilczyński, W. and Malik, P., (2008). *$\mathcal{J}$ and $\mathcal{J}^*$-convergence of double sequences.* Mathematica Slovaca, **58**(5), 605–620.

Dündar, E., Ulusu, U. and Pancaroğlu, N., (2016). *Strongly I_2-convergence and I_2-lacunary Cauchy double sequences of sets.* The Aligarh Bulletin of Mathematics, **35**(1-2), 1-15.

Dündar, E., Ulusu, U. and Aydın, B., (2017). *I_2-lacunary statistical convergence of double sequences of sets.* Konuralp Journal of Mathematics, **5**(1), 1-10.

Dündar, E., Ulusu, U. and Nuray, F., *On asymptotically ideal invariant equivalence of double sequences*. (In review).

Fast, H., (1951). *Sur la convergence statistique*. Colloquium Mathematicum, **2**, 241-244.

Khan, V. A. and Khan, N., (2013). *On Some $\mathcal{J}$-Convergent Double Sequence Spaces Defined by a Modulus Function*, Engineering, **5**, 35–40.

Kılınç, G. and Solak, İ., (2014). *Some Double Sequence Spaces Defined by a Modulus Function*. General Mathematics Notes, **25**(2), 19–30.

Kişi, Ö. and Nuray, F., (2013). *A new convergence for sequences of sets.* Abstract and Applied Analysis, Article ID 852796

Kişi Ö., Gümüş H. and Nuray F., (2015).*$\mathcal{J}$-Asymptotically lacunary equivalent set sequences defined by modulus function.* Acta Universitatis Apulensis, **41**, 141-151.

Kostyrko P., Šalát T. and Wilczyński W., (2000).*$\mathcal{J}$-Convergence*. Real Analysis Exchange, **26**(2), 669-686.

Kumar V. and Sharma A., (2012). *Asymptotically lacunary equivalent sequences defined by ideals and modulus function.* Mathematical Sciences, **6**(23), 5 pages.

Lorentz G., (1948). *A contribution to the theory of divergent sequences*. Acta Mathematica, **80**, 167-190.

Maddox J., (1986). *Sequence spaces defined by a modulus.* Mathematical Proceedings of the Cambridge Philosophical Society, **100**, 161-166.

Marouf, M., (1993). *Asymptotic equivalence and summability*. Int. J. Math. Math. Sci., **16**(4), 755-762.

Mursaleen, M.,(1983). *Matrix transformation between some new sequence spaces*. Houston Journal of Mathematics, **9**, 505-509.

Mursaleen, M. and Edely, O. H. H., (2009). *On the invariant mean and statistical convergence*. Applied Mathematics Letters, **22**(11), 1700-1704.

Nakano H., (1953). *Concave modulars*. Journal of the Mathematical Society Japan, **5** ,29-49.

Nuray, F. and Savaş, E., (1994). *Invariant statistical convergence and A-invariant statistical convergence*. Indian Journal of Pure and Applied Mathematics, **25**(3), 267-274.

Nuray, F., Gök, H. and Ulusu, U., (2011). $\mathcal{I}_{\sigma}$*-convergence*. Mathematical Communications, **16**, 531-538.

Nuray F. and Rhoades B. E., (2012). *Statistical convergence of sequences of sets*. Fasiciculi Mathematici, **49** , 87-99.

Nuray, F., Ulusu, U. and Dündar, E., (2016). *Lacunary statistical convergence of double sequences of sets.* Soft Computing, **20**, 2883-2888.

Pancaroğlu Akın, N. and Dündar, E.,(2018*). Asymptotically $\mathcal{I}$-Invariant Statistical Equivalence of Sequences of Set Defined by a Modulus Function.* AKU Journal of Science Engineering, **18**(2), 477–485.

Pancaroğlu Akın, N., Dündar, E., and Ulusu, U., (2018*).Asymptotically $\mathcal{I}_{\sigma\theta}$-statistical Equivalence of Sequences of Set Defined By A Modulus Function.* Sakarya University Journal of Science, **22**(6), x-x. doi: 10.16984/saufenbilder

Pancaroğlu Akın, N., *Wijsman lacunary $\mathcal{I}_2$-invariant convergence of double sequences of sets.* (In review).

Pancaroğlu, N. and Nuray, F., (2013). *Statistical lacunary invariant summability.* Theoretical Mathematics and Applications, **3**(2), 71-78.

Pancaroğlu N. and Nuray F., (2013). *On Invariant Statistically Convergence and Lacunary Invariant Statistically Convergence of Sequences of Sets*. Progress in Applied Mathematics, **5**(2), 23-29.

Pancaroğlu N. and Nuray F. and Savaş E., (2013). *On asymptotically lacunary invariant statistical equivalent set sequences*. AIP Conf. Proc. **1558**(780) http://dx.doi.org/10.1063/1.4825609

Pancaroğlu N. and Nuray F., (2014*). Invariant Statistical Convergence of Sequences of Sets with respect to a Modulus Function*. Abstract and Applied Analysis, Article ID 818020, 5 pages.

Patterson, R. F., (2003). *On asymptotically statistically equivalent sequences. Demostratio Mathematica*, **36**(1), 149-153.

Pehlivan S., and Fisher B., (1995).*Some sequences spaces defined by a modulus*. Mathematica Slovaca, **45**, 275-280.

Raimi, R. A., (1963). *Invariant means and invariant matrix methods of summability*. Duke Mathematical Journal, **30**(1), 81-94.

Savaş, E., (1989). *Some sequence spaces involving invariant means.* Indian Journal of Mathematics, **31**, 1-8.

Savaş, E., (1989*). Strongly σ-convergent sequences*. Bulletin of Calcutta Mathematical Society, **81**, 295-300.

Savaş, E. and Nuray, F., (1993). *On σ-statistically convergence and lacunary σ-statistically convergence*. *Mathematica Slovaca*, **43**(3), 309-315.

Savaş, E., 2013. *On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences*. Advances in Difference Equations, **111**(2013), 7 pages. doi:10.1186/1687-1847-2013-111

Schaefer, P., (1972). *Infinite matrices and invariant means*. Proceedings of the American Mathe-matical Society, **36**, 104-110.

Schoenberg I. J., (1959). *The integrability of certain functions and related summability methods*. American Mathematical Monthly, **66**, 361-375.

Tortop, Ş. and Dündar, E., (2018). *Wijsman I2-invariant convergence of double sequences of sets*. Journal of Inequalities and Special Functions,(In press).

Ulusu U. and Nuray F., (2013). *On asymptotically lacunary statistical equivalent set sequences*. Journal of Mathematics, Article ID 310438, 5 pages.

Ulusu, U. and Dündar, E., (2014). *I-lacunary statistical convergence of sequences of sets*. Filomat, **28**(8), 1567-1574.

Ulusu, U., Dündar, E. and Nuray, F., (2018). *Lacunary I_2-invariant convergence and some properties*. International Journal of Analysis and Applications, **16**(3), 317-327.

Ulusu U. and Gülle E., (2019). *Asymptotically* $\mathcal{I}_{\sigma}$*-equivalence of sequences of sets*. (In press).

Wijsman R. A., (1964). *Convergence of sequences of convex sets, cones and functions*. Bulletin American Mathematical Society, **70**, 186-188.

Wijsman R. A., (1966). *Convergence of Sequences of Convex sets, Cones and Functions II.* Transactions of the American Mathematical Society, **123**(1) , 32-45.