Wijsman and Hausdorff Statistical Convergence of Order α for Double Set Sequences

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Abstract

The concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951), and later reintroduced by Schoenberg (1959) independently. Then, many researchers have studied on this concept until recently (see Fridy 1985; Savaş 2002; Šalát 1980; Tripathy 1998). The order of statistical convergence of a single sequence of numbers was given by Gadjiev and Orhan (2001). Then, the concepts of statistical convergence of order α was studied by Çolak (2010) and Çolak and Bektaş (2011).

In 1900, Pringsheim introduced the concept of convergence for double sequences. Recently, Mursaleen and Edely (2003) extended this concept to statistical convergence. More developments on double sequences can be found in Çakallı and Savaş (2010), Mohiuddine et al. (2012) and Bhunia et al. (2012). Very recently, the concept of statistical convergence of order α for double sequences was studied by Savaş (2013) and Çolak and Altın (2013).

The concept of convergence for number sequences was transferred to the concepts of convergence for set sequences by many authors. In this study, the concepts of Wijsman convergence and Hausdorff convergence which are two of these transfers are considered (see Baronti and Papini 1986; Beer 1985, 1994; Wijsman 1964). Nuray and Rhoades (2012) extended the concepts of Wijsman convergence and Hausdorff convergence to statistical convergence for set sequences and gave some basic theorems. Very recently, the concept of Wijsman I-statistical convergence of order α was studied by Savaş (2015) and Şengül and Et (2017).

Nuray et al. (2014) introduced the concept of Wijsman convergence for double set sequences. Then, the concept of Hausdorff convergence for double set sequences was studied by Sever et al. (2015). Also, the concepts of Wijsman statistical convergence and Hausdorff statistical convergence were studied by Nuray et al. (2019) and Talo et al. (2016), respectively.

In this study, we introduce the concepts of Wijsman statistical convergence of order α and Hausdorff statistical convergence of order α for double set sequences. Also, we investigate some properties of these concepts and examine the relationship between them.

Keywords: Statistical convergence, double sequence, order α, Wijsman convergence, Hausdorff convergence, Set sequences.

Öz

İstatistiksel yakınsaklık kavramı Steinhaus (1951) ve Fast (1951) tarafından tanıtılmış, ve daha sonra bağımsız olarak Schoenberg (1959) tarafından yeniden tanımlanmıştır. Yakın zamana kadar pek çok araştırmacı da bu kavram üzerine çalışmıştır (bkz Fridy 1985; Savaş 2002; Šalát 1980; Tripathy 1998). Reel sayı dizilerinin istatistiksel yakınsaklık mertebesi Gadjiev ve Orhan (2001) tarafından verilmiştir. Daha sonra Çolak (2010) ve Çolak ve Bektaş (2011) tarafından α. mertebeden istatistiksel yakınsaklık kavramı çalışılmıştır.

Pringsheim 1900 de çift diziler için yakınsaklık kavramını tanıtmıştır. Mursaleen ve Edely (2003) bu kavramı istatistiksel yakınsaklığa genişletmiştir. Çift diziler üzerine yapılan pek çok çalışma Çakallı ve Savaş (2010), Mohiuddine vd. (2012) ve Bhunia vd. (2012) de bulunabilir. Son zamanlarda, çift diziler için α. mertebeden istatistiksel yakınsaklık kavramı Savaş (2013) ve Çolak ve Altın (2013) tarafından çalışılmıştır.

Sayı dizileri için yakınsaklık kavramı pek çok araştırmacı tarafından küme dizileri için yakınsaklık kavramlarına aktarılmıştır. Bu çalışmada, küme dizileri için Wijsman ve Hausdorff yakınsaklık kavramları ele alınmıştır (bkz Baronti ve Papini 1986; Beer 1985, 1994; Wijsman 1964). Nuray ve Rhoades (2012) küme dizileri için Wijsman ve Hausdorff istatistiksel yakınsaklık kavramlarını çalışmış ve bazı temel teoremleri vermiştir. Son zamanlarda, α. mertebeden Wijsman I-istatistiksel yakınsaklık kavramı Savaş (2015) ve Şengül ve Et (2017) tarafından çalışılmıştır.

Nuray vd. (2014) çift küme dizileri için Wijsman yakınsaklık kavramını tanıtmıştır. Daha sonra Sever vd. (2015) tarafından çift küme dizileri için Hausdorff yakınsaklık kavramı çalışılmıştır. Ayrıca, Wijsman istatistiksel yakınsaklık ve Hausdorff istatistiksel yakınsaklık kavramları sırasıyla, Nuray vd. (2019) ve Talo vd. (2016) tarafından incelenmiştir.

Bu çalışmada, çift küme dizileri için α. mertebeden Wijsman istatistiksel yakınsaklık ve α. mertebeden Hausdorff istatistiksel yakınsaklık kavramları tanıtılmıştır. Ayrıca, bu kavramların bazı özellikleri araştırılmış ve bunlar arasındaki ilişki incelenmiştir.

Anahtar Kelimeler: İstatistiksel yakınsaklık, çift dizi, α. mertebe, Wijsman yakınsaklık, Hausdorff yakınsaklık, küme dizisi.

Introduction and Basic Concepts

Firstly, we recall the basic concepts that need for a good understanding of our study (see Baronti and Papini 1986; Beer 1985; Mursaleen and Edely 2003; Nuray et al. 2014, 2019; Pringsheim 1900; Sever et al. 2015; Talo et al. 2016; Wijsman 1964).

A double sequence (x_{ij}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$, whenever $i, j > N_\varepsilon$.

A double sequence (x_{ij}) is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{(i,j): i \le m, j \le n, \quad |x_{ij} - L| \ge \varepsilon\right\}\right| = 0.$$

Let X be any non-empty set. The function $f: \mathbb{N} \to P(X)$ is defined by $f(i) = U_i \in P(X)$ for each $i \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{U_i\} = (U_1, U_2, \dots)$, which is the range's elements of f, is said to be set sequences.

Let (X, d) be a metric space. For any point $x \in X$ and any non-empty subset U of X, the distance from x to U is defined by

$$\rho(x, U) = \inf_{u \in U} d(x, u).$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \rho(x, U_{ij}) = \rho(x, U).$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Hausdorff convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \sup_{x \in X} |\rho(x, U_{ij}) - \rho(x, U)| = 0.$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman statistically convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j): i \le n, j \le m, \quad |\rho(x, U_{ij}) - \rho(x, U)| \ge \varepsilon\}| = 0.$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Hausdorff statistically convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j): i \le n, j \le m, \quad \sup_{x \in X} |\rho(x, U_{ij}) - \rho(x, U)| \ge \varepsilon\}| = 0.$$

From now on, for short, we use $\rho_x(U)$ and $\rho_x(U_{ij})$ instead of $\rho(x, U)$ and $\rho(x, U_{ij})$, respectively.

Method

In the proofs of the theorems obtained from this study, the following proof methods, which are frequently used in mathematics, have been used as needed:

i. Direct proof method,
ii. Inverse situation proof method,

iii. Method of non-finding (contradiction),

iv. Induction method.

Results

In this section, we introduce the concepts of Wijsman statistical convergence of order α and Hausdorff statistical convergence of order α for double set sequences. Also, we investigate some properties of these concepts and examine the relationship between them.

Definition:

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Wijsman statistically convergent of order α to U or $W(S_2^\alpha)$-convergent to U if for each $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} |\{(i,j): i \leq m, j \leq n, \qquad |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\}| = 0.$$

In this case, we write $U_{ij} \stackrel{W(S_2^\alpha)}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(W(S_2^\alpha))$.

The class of all $W(S_2^\alpha)$-convergent sequences will be denoted by simply $W(S_2^\alpha)$.

Example:

Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$\{U_{ij}\} = \begin{cases} \{(x,y) \in \mathbb{R}^2: (x-i)^2 + (y+j)^2 = 4\} & , \text{ if } i \text{ and } j \text{ are sequare integer} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent of order α to the set $U = \{(1,1)\}$.

For $\alpha = 1$, the concept of $W(S_2^\alpha)$-convergence coincides with the concept of Wijsman statistical convergence ($W(S_2)$-convergence) for double set sequences in Nuray et al. (2019).

Theorem:

If $0 < \alpha \leq \beta \leq 1$, then $W(S_2^\alpha) \subset W(S_2^\beta)$.

If we take $\beta = 1$ in above Theorem, then we get $W(S_2^\alpha) \subset W(S_2)$.

Definition:

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Hausdorff statistically convergent of order α to U or $H(S_2^\alpha)$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} |\{(i,j): i \leq m, j \leq n, \quad \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\}| = 0.$$

In this case, we write $U_{ij} \overset{H(S_2^\alpha)}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(H(S_2^\alpha))$.

The class of all $H(S_2^\alpha)$-convergent sequences will be denoted by simply $H(S_2^\alpha)$.

For $\alpha = 1$, the concept of $H(S_2^\alpha)$-convergence coincides with the concept of Hausdorff statistical convergence ($H(S_2)$-convergence) for double set sequences in Talo et al. (2016).

Theorem:

If $0 < \alpha \leq \beta \leq 1$, then $H(S_2^\alpha) \subset H(S_2^\beta)$.

If we take $\beta = 1$ in above Theorem, then we get $H(S_2^\alpha) \subset H(S_2)$.

Theorem:

Let $0 < \alpha \leq 1$. If a double sequence $\{U_{ij}\}$ is Hausdorff statistically convergent of order α to U, then the sequence is Wijsman statistically convergent of order α to U.

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